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# Akhiezer's Orthogonal Polynomials and Bernstein–Szegő Method for a Circular Arc

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Orthogonal polynomials theory on a circular arc was apparently first developed by N. I. Akhiezer, who announced his asymptotic formulas for orthogonal polynomials on and off the support of orthogonality measure in a short note in *Doklady AN SSSR*. We present here a rigorous exposition of Akhiezer's result and outline some mild generalizations of the theory. 

## 1. INTRODUCTION

The theory of orthogonal polynomials on the unit circle was created by G. Szegő in the early twenties (cf. [15]) and developed afterwards by G. Freud and Ja. L. Geronimus. It concerns polynomial system $\varphi_n(\mu, z)$ which satisfy

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(\mu, e^{i\vartheta})\,\overline{\varphi_m(\mu, e^{i\vartheta})}\,d\mu = \delta_{m,n}, \qquad m, n = 0, 1, 2, ...,$$

where

$$\varphi_n(\mu, e^{i\vartheta}) = \kappa_n(\mu)\,z^n + \text{lower degree terms}, \qquad \kappa_n(\mu) > 0,$$

and $\mu$ is a positive Borel measure in $[0, 2\pi)$ with infinite support. The monic orthogonal polynomials $\Phi_n = \kappa_n^{-1}(\mu)\,\varphi_n = z^n + \cdots$ are also of great importance, as they do not alter under multiplication of the measure $\mu$ by a positive constant. What is more to the point, the measure $\mu$ and the whole system $\varphi_n$ is fully determined by the sequence of complex numbers $\{\Phi_n(0)\}_{n=0}^{\infty}$, which are usually called *reflection coefficients*.



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Over a period of nearly fifty years the theory was confined primarily to a certain class of measures (know now as *Szegő class*) with the property

$$\log \mu' \in L^1 \Leftrightarrow \sum_{k=1}^{\infty} |\Phi_k(\mu, 0)|^2 < \infty, \tag{1}$$

where $\mu'$ is the Radon–Nikodym derivative of $\mu$ with respect to Lebesgue measure. One of the highlights of this theory is the Szegő asymptotic formula for orthonormal polynomials (cf. [8, Theorem 3.4])

$$\varphi_n(\mu, z) = \frac{z^n}{\bar{D}(\mu', 1/z)} (1 + o(1)), \qquad n \to \infty, \quad |z| > 1 \tag{2}$$

uniformly on compact subsets of the domain $|z| > 1$ on the Riemann sphere. Here $D(\mu', z)$ is the Szegő function, i.e., an outer function from $H^2$ in the unit disk, corresponding to the limit values $\sqrt{\mu'}$. Under more restrictive assumptions on the function $\mu'$ (cf. [6, Corollary 1.2, p. 153]) and (93) below) the asymptotic formula holds uniformly on the unit circle as well.

A truly major step towards extending Szegő's theory was made by E. A. Rahmanov [13], who replaced the logarithmic integrability condition by the much weaker one, $\mu' > 0$ almost everywhere. In [10, 11] P. Nevai with his collaborators carried over a considerable part of Szegő's theory to an even more extensive class of measures (now known as Nevai class), wherein $\lim_{n \to \infty} \Phi_n(\mu, 0) = 0$.

It is appropriate to mention here an old result by Geronimus (cf. [4, Theorem 19.1]), according to which a (closed) support of a measure from the Nevai class is the whole interval $[0, 2\pi)$. It means that every measure, having a proper subset, for instance, an interval $[a, b] \subset [0, 2\pi)$, as its support, lies outside the Nevai class. Surprisingly enough, the results concerning the orthogonal polynomials of such type were apparently first proved (announced, to be exact) by N. I. Akhiezer in the short note [1] in *Doklady AN SSSR* as far back as 1960. Being highly quoted, this paper has not been fully appraised by experts due to the lack of transparent proofs of the statements therein. Although the theory of orthogonal polynomials for general arcs in the complex plane (and even systems of arcs), including asymptotic relations on and off the support of measure, has been developed vastly at the moment due to H. Widom [16] and V. Kaliaguine [9] (see also [12] for the special case of the circular arcs and [2] for Rahmanov's theory on a circular arc), in my opinion, the results and especially the method applied in [1] are still worth studying.

Our principal goal is to present a rigorous exposition of Akhiezer's results, providing the reader with all necessary details, and to outline possible extensions of the method. The main object under consideration is a weight function $W$ on the arc

$$\Delta_\alpha \stackrel{\text{def}}{=} \{e^{i\vartheta} : \alpha \leqslant \vartheta \leqslant 2\pi - \alpha\}, \qquad 0 < \alpha < \pi, \tag{3}$$

which satisfies

$$\rho(e^{i\vartheta}) \stackrel{\text{def}}{=} W(e^{i\vartheta}) \frac{\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}{\sin(\vartheta/2)} \in C(\Delta_\alpha), \qquad 0 < l \leqslant \rho(e^{i\vartheta}) \leqslant L < \infty \tag{4}$$

(i.e., $W$ is positive and continuous on $\Delta_\alpha$ and has square root singularities at both endpoints), and a corresponding system of orthonormal polynomials $\varphi_n$.

The paper is organized as follows. In Sections 2 and 3 we consider a special class of polynomials orthogonal on $\Delta_\alpha$. Section 4 contains an account of the Bernstein–Szegő approximation method applied to the arc (3). The asymptotic formulas for orthogonal polynomials, corresponding to the weight functions of the form (4) on the arc $\Delta_\alpha$, are given in Section 5.

## 2. AKHIEZER'S ORTHOGONAL POLYNOMIALS

*Special Conformal Mapping.* Given a positive number $0 < \alpha < \pi$, set

$$\eta = \frac{\pi - \alpha}{4}, \qquad \beta = i \tan \eta = -\bar{\beta} \qquad (|\beta| < 1) \tag{5}$$

and consider a rational function $z = h(v)$ in the unit disk $\mathbb{D} = \{|v| < 1\}$:

$$z = h(v) \stackrel{\text{def}}{=} \frac{(v-\beta)(\beta v - 1)}{(v+\beta)(\beta v + 1)} = -\frac{v-\beta}{1-\bar{\beta}v}\,\frac{1+\bar{\beta}v}{v+\beta} = \frac{v-\beta}{v+\beta}\,\frac{v-\beta^{-1}}{v+\beta^{-1}}. \tag{6}$$

The following properties of $h(v)$ are of particular interest.

(1) $h(v)$ is analytic in $\mathbb{D} \setminus \{-\beta\}$ and has a simple pole at the point $-\beta$;

(2) "individual values": $h(0) = 1$, $h(\beta) = 0$, $h(\pm i) = -1$,

$$h(1) = -\left(\frac{1-\beta}{1+\beta}\right)^2 = -e^{-4i\eta} = e^{i\alpha}, \qquad h(-1) = e^{-i\alpha},$$

$|h(x)| = 1$ for real $x$;

(3) $h(v) = -b_2(v)/b_1(v)$, where the $b_i(v)$ are Blaschke factors, $i = 1, 2$, so that $|h(e^{i\omega})| = 1$;

(4) "symmetry": the function $h(v)$ is defined on the whole complex plane and satisfies

$$h(v^{-1}) = h(v), \qquad h\left(\frac{1}{\bar v}\right) = (\overline{h(v)})^{-1}. \tag{7}$$

In particular,

$$h(e^{-i\omega}) = h(e^{i\omega}), \tag{8}$$

that is, the conjugate points on the unit circle are being stuck together;

(5) Let $v = e^{i\omega}$. The equation $e^{i\vartheta} = h(e^{i\omega})$ can be solved for $\omega$ explicitly:

$$e^{2i\omega} - (\beta + \beta^{-1})\, e^{i\omega} + 1 = e^{i\vartheta}(e^{2i\omega} + (\beta + \beta^{-1})\, e^{i\omega} + 1),$$

$$e^{2i\omega} - i(\beta + \beta^{-1}) \cot \frac{\vartheta}{2}\, e^{i\omega} + 1 = 0.$$

If $0 < \omega < \pi$, then

$$e^{i\omega} = \frac{\tan(\alpha/2)}{\tan(\vartheta/2)} + i\sqrt{1 - \left(\frac{\tan(\alpha/2)}{\tan(\vartheta/2)}\right)^2}$$

and hence

$$\cos\omega = \frac{\tan(\alpha/2)}{\tan(\vartheta/2)}, \qquad \sin\omega = \sqrt{1 - \left(\frac{\tan(\alpha/2)}{\tan(\vartheta/2)}\right)^2} = \frac{\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}{\cos(\alpha/2)\sin(\vartheta/2)}. \tag{9}$$

Thus, while the point $e^{i\omega}$ runs over the unit circle, the point $e^{i\vartheta}$ sweeps the arc $\Delta_\alpha$ twice;

(6) $$h'(v) = 2(\beta + \beta^{-1}) \frac{v^2 - 1}{(v + \beta)^2\, (v + \beta^{-1})^2} \neq 0, \qquad v \in \mathbb{D}, \tag{10}$$

that is, $h(v)$ maps conformally the unit disk onto the domain $\mathbb{C}\backslash\Delta_\alpha$.

It is reasonable to handle the arc $\Delta_\alpha$ as a cut on the complex plane with two borders: an interior border $\Delta_\alpha^-$ and an exterior border $\Delta_\alpha^+$. When the point $v$ tends to $e^{i\omega_0}$, $0 < \omega_0 < \pi$, the image $h(v)$ goes to $e^{i\vartheta_0} \in \Delta_\alpha^-$ from the inside of the unit disk. If the point $v$ tends to $e^{-i\omega_0}$, then $h(v)$ goes to $e^{i\vartheta_0} \in \Delta_\alpha^+$ from the outside of the unit disk. Indeed, putting $\xi = (\tan\eta)^{-1} - \tan\eta$, we have for $v = re^{i\omega_0}$, $0 < r < 1$,

$$h(v)=\frac{v^2+iv\xi+1}{v^2-iv\xi+1},$$

$$|h(v)|^2-1=\frac{|v^2+iv\xi+1|^2-|v^2-iv\xi+1|^2}{|v^2-iv\xi+1|^2}=-\frac{4r(1-r^2)\,\xi\,\sin\omega_0}{|v^2-iv\xi+1|^2},$$

as claimed. Hence the upper (resp. lower) semicircle corresponds to the interior (resp. exterior) border of the cut $\Delta_\alpha$.

Consider an auxiliary conformal mapping $v(x)\colon \mathbb{D}_- \to \mathbb{D}$, $\mathbb{D}_- = \{|w|>1\}$ such that $v(\infty)=-\beta$:

$$v(w)\overset{\text{def}}{=}\frac{i-\beta w}{w+i\beta}. \tag{11}$$

The converse mapping is given by

$$w(v)=i\,\frac{1+\bar{\beta}v}{v+\beta}=i\,\frac{1-\beta v}{v+\beta}. \tag{12}$$

The composition $z=z(w)=h(v(w))$ maps $\mathbb{D}_-$ onto $\mathbb{C}\backslash\Delta_\alpha$:

$$z=\frac{\cos(\alpha/2)\,w^2-w}{w-\cos(\alpha/2)}=\gamma w+\sum_{k=0}^{\infty} d_k w^{-k}, \tag{13}$$

where

$$\gamma\overset{\text{def}}{=}\lim_{w\to\infty}\frac{h(v(w))}{w}=\lim_{v\to-\beta}\frac{(v-\beta)(\beta v-1)}{(v+\beta)(\beta v+1)}\,\frac{v+\beta}{i(1+\bar{\beta}v)}$$

$$=-\frac{2\beta i}{1-\beta^2}=\cos\frac{\alpha}{2} \tag{14}$$

is the transfinite diameter of $\Delta_\alpha$.

The mapping $w=w(z)\colon \mathbb{C}\backslash\Delta_\alpha\to\mathbb{D}_-$ can be easily found from (13)

$$w(z)=\frac{z+1+R(z)}{2\gamma},$$

$$R(z)\overset{\text{def}}{=}\sqrt{(z+1)^2-4z\cos^2\frac{\alpha}{2}}=\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}, \tag{15}$$

where that branch of the square root is chosen for which $R(0)=1$. To calculate the interior (resp. exterior) boundary values $w_-(e^{i\vartheta})$ (resp. $w_+(e^{i\vartheta})$), notice that $w_\pm(-1)=\mp 1$ and hence

$$w_{\pm}(e^{i\vartheta}) = \frac{e^{i\vartheta} + 1 \pm 2ie^{i(\vartheta/2)} \sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}{2\gamma}$$

$$= \frac{e^{i(\vartheta/2)}}{\gamma} \left\{ \cos\frac{\vartheta}{2} \pm i \sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{\vartheta}{2}} \right\}. \tag{16}$$

Putting

$$\cos\lambda = \frac{\cos(\vartheta/2)}{\cos(\alpha/2)}, \qquad 0 \leqslant \lambda \leqslant \pi, \tag{17}$$

we finally get

$$w_{\pm}(e^{i\vartheta}) = \exp\left\{ i\left(\frac{\vartheta}{2} \pm \lambda\right)\right\}. \tag{18}$$

The function $h(v)$ on the unit circle generates the change of variables formula, which plays a crucial role throughout the paper:

$$\int_0^{\pi} \tilde{f}(e^{i\omega})\, d\omega = \int_{\alpha}^{2\pi-\alpha} f(e^{i\vartheta}) \frac{h(e^{i\omega})}{h'(e^{i\omega})\, e^{i\omega}}\, d\vartheta$$

$$= \int_{\alpha}^{2\pi-\alpha} \frac{f(e^{i\vartheta}) \sin(\alpha/2)}{2\sin(\vartheta/2) \sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}\, d\vartheta, \tag{19}$$

where $\tilde{f}(e^{i\omega}) \overset{\text{def}}{=} f(h(e^{i\omega}))$. The weight function

$$W(e^{i\vartheta}; 1) \overset{\text{def}}{=} \frac{\sin(\alpha/2)}{2\sin(\vartheta/2)\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}, \tag{20}$$

which occurs in (19)), may be regarded as the first kind Chebyshev weight function for the circular arc $\Delta_\alpha$.

*Akhiezer's Polynomials and Special Weight Functions.* We proceed with the following

LEMMA 1. *For nonnegative integers $k$, $l$ the function*

$$P_n(z) = w^k(v)\, w^l\left(\frac{1}{v}\right) + w^l(v)\, w^k\left(\frac{1}{v}\right), \qquad z = h(v)$$

*is a polynomial of degree $n = \max(k, l)$ with a positive leading coefficient.*

*Proof.* It is quite clear from (6) and (12) that

$$w(v)\, w\left(\frac{1}{v}\right) = h(v) = z, \qquad w(v) + w\left(\frac{1}{v}\right) = \frac{z+1}{\gamma}. \tag{21}$$

The assertion of the lemma now follows by induction from identity

$$w^{n+1}(v)+w^{n+1}\left(\frac{1}{v}\right)=\left(w^{n}(v)+w^{n}\left(\frac{1}{v}\right)\right)\left(w(v)+w\left(\frac{1}{v}\right)\right) \\ -w(v)\,w\left(\frac{1}{v}\right)\left(w^{n-1}(v)+w^{n-1}\left(\frac{1}{v}\right)\right).$$

and (21). ∎

Let $\Omega(v)$ be a rational function, real on the real line. Set

$$\varphi_n(z;\Omega)\overset{\text{def}}{=}K_n\left\{\frac{\Omega(1/v)}{1-\beta v}\,w^{n}(v)+\frac{v\Omega(v)}{v-\beta}\,w^{n}\left(\frac{1}{v}\right)\right\},\qquad z=h(v),\tag{22}$$

where $K_n$ is a nonzero complex number. We want to specify the functions $\Omega$, which generate $n$th degree polynomials of $z$ $\varphi_n(z;\Omega)$ at least for large enough $n\geqslant n_0(\Omega)$. The examples below show that the set of such functions is not empty.

EXAMPLE 1. Let $\Omega(v)=1$. Since

$$\frac{v}{v-\beta}=\frac{i\beta}{1+\beta^2}\,w^{-1}\left(\frac{1}{v}\right)+\frac{1}{1+\beta^2},\tag{23}$$

the functions $\varphi_n$ in (22) are polynomials of $z$ for $n\geqslant 1$ in light of Lemma 1.

EXAMPLE 2. Let $\Omega(v)=v^{-1}$. Much as in Example 1, we see that the $\varphi_n$ in (22) are polynomials of $z$ for $n\geqslant 1$.

EXAMPLE 3. Let $\Omega(v)=\Omega_{\pm}(v)\overset{\text{def}}{=}(v\pm 1)^{-1}$, so that

$$K_n^{-1}\varphi_n(z;\Omega_{\pm})=\frac{v}{1\pm v}\left\{\frac{1}{1-\beta v}\,w^{n}(v)\pm\frac{1}{v-\beta}\,w^{n}\left(\frac{1}{v}\right)\right\}.$$

The conclusion (with $n_0(\Omega_{\pm})=1$) follows by induction from the equality

$$K_n^{-1}\varphi_n(z;\Omega_{\pm})=K_{n-1}^{-1}\varphi_{n-1}(z;\Omega_{\pm})\left(w(v)+w\left(\frac{1}{v}\right)\right) \\ -K_{n-2}^{-1}\varphi_{n-2}(z;\Omega_{\pm})\,w(v)\,w\left(\frac{1}{v}\right)$$

for $n\geqslant 3$ and direct (though lengthy) calculation for $n=1, 2$.

EXAMPLE 4. Let

$$\Omega(v)=\Omega_q(v)\overset{\text{def}}{=}\frac{P(v)}{(v^2-\beta^2)^q},$$

where $P$ is a real polynomial of degree $r=\deg P\leqslant 2q$ and $P(\pm\beta)\neq 0$. Decompose $P=P_1P_2$ in such a way that $r_i=\deg P_i\leqslant q$, $i=1,2$. Then

$$\begin{aligned}\Omega_q(v)&=\frac{P_1(v)}{(v+\beta)^q}\,\frac{P_2(v)}{(v-\beta)^q}=\sum_{j=0}^{r_1}\frac{P_{j1}}{(v+\beta)^{q-j}}\sum_{j=0}^{r_2}\frac{P_{j2}}{(v-\beta)^{q-j}}\\&=Q_1\left(\frac{1}{v+\beta}\right)Q_2\left(\frac{1}{v-\beta}\right).\end{aligned}$$

As $P_{01}=P_1(-\beta)\neq 0$, $P_{02}=P_2(\beta)\neq 0$, the polynomials $Q_i$ have degree exactly $q$ for $i=1,2$. But

$$\frac{1}{v+\beta}=Aw(v)+B,\qquad \frac{1}{v-\beta}=\bar{A}w^{-1}\left(\frac{1}{v}\right)+\bar{B},$$

and (23) implies now

$$\frac{v}{v-\beta}\,\Omega_q(v)=Q_3(w(v))\,Q_4\left(w^{-1}\left(\frac{1}{v}\right)\right),\qquad \deg Q_3=q,\quad \deg Q_4=q+1.$$

Thus by Lemma 1 the functions

$$\varphi_n(z;\Omega_q)=Q_5(w(v))\,Q_6\left(w\left(\frac{1}{v}\right)\right)+Q_5\left(w\left(\frac{1}{v}\right)\right)Q_6(w(v))$$

are polynomials of $z$ for $n\geqslant n_0(\Omega_q)=q+1$.

It is obvious that any linear combination of the functions $\Omega$ from the above examples generates by means of (22) polynomials of $z$ for large enough $n$. We shall impose two more restrictions on the function $\Omega$, presuming that $\Omega(v)\neq 0$ for $|v|\geqslant 1$, including infinity, and $\Omega(x)=\overline{\Omega(x)}$ for real $x$. Let $\mathscr{M}$ denote such a class of functions. In other words,

$$\mathscr{M}\overset{\text{def}}{=}\left\{\Omega(v)=\frac{P(v)}{v^{\varepsilon_0}(v-1)^{\varepsilon_-}(v+1)^{\varepsilon_+}(v^2-\beta^2)^q},\ \begin{array}{l}\varepsilon_0,\varepsilon_\pm=0\text{ or }1,\\ \deg P=2q+\varepsilon_0+\varepsilon_-+\varepsilon_+\end{array}\right\},\tag{24}$$

where $P$ is a real polynomial, which has no zeros outside $\mathbb{D}$. The function $\Omega$ is thereafter assumed to belong to $\mathscr{M}$.

To make sure that $\varphi_n$ in (22) is a polynomial of exactly degree $n$, let us calculate its leading coefficient

$$\kappa_n(\Omega) \stackrel{\text{def}}{=} \lim_{z\to\infty} \frac{\varphi_n(z;\Omega)}{z^n}$$

$$= \lim_{v\to -\beta} K_n \left\{ \frac{v\Omega(v)}{v-\beta} w^{-n}(v) + \frac{\Omega(1/v)}{1-\beta v} w^{-n}\left(\frac{1}{v}\right) \right\}$$

$$= K_n \Omega\left(-\frac{1}{\beta}\right) \frac{1+\sin(\alpha/2)}{2\sin(\alpha/2)} \gamma^{-n} \neq 0 \tag{25}$$

for $n \geqslant n_0(\Omega)$. In what follows, we always take $K_n = K_n(\Omega)$ to satisfy

$$\arg K_n(\Omega) = \arg \Omega\left(\frac{1}{\beta}\right), \tag{26}$$

so that $\kappa_n(\Omega) > 0$.

The main feature of the polynomials $\varphi_n$ is their orthogonality on the arc $\Delta_\alpha$ with respect to the special weight function

$$W(e^{i\vartheta};\Omega) \stackrel{\text{def}}{=} \frac{W(e^{i\vartheta};1)}{|\Omega(e^{i\omega})|^2}$$

$$= \frac{\sin(\alpha/2)}{2\sin(\vartheta/2)\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}\,|\Omega(e^{i\omega})|^2}, \qquad e^{i\vartheta} = h(e^{i\omega}), \tag{27}$$

which is well defined due to the property $|\Omega(e^{i\omega})| = |\Omega(e^{-i\omega})|$. Indeed, by the change of variables formula we have for $m = 0, 1, ..., n-1$ and $n \geqslant n_0(\Omega)$

$$\int_\alpha^{2\pi-\alpha} K_n^{-1} \varphi_n(e^{i\vartheta};\Omega)\, e^{-im\vartheta} W(e^{i\vartheta};\Omega)\, d\vartheta$$

$$= \int_0^\pi \left\{ \frac{\Omega(e^{-i\omega})}{1-\beta e^{i\omega}} w^n(e^{i\omega}) + \frac{e^{i\omega}\Omega(e^{i\omega})}{e^{i\omega}-\beta} w^n(e^{-i\omega}) \right\} \frac{w^{-m}(e^{i\omega})\, w^{-m}(e^{-i\omega})\, d\omega}{|\Omega(e^{i\omega})|^2}$$

$$= i^{n-2m} \int_0^\pi \left(\frac{1-\beta e^{i\omega}}{e^{i\omega}+\beta}\right)^{n-m} \left(\frac{1+\beta e^{i\omega}}{e^{i\omega}-\beta}\right)^m \frac{d\omega}{(1-\beta e^{i\omega})\,\Omega(e^{i\omega})}$$

$$+ i^{n-2m} \int_0^\pi \left(\frac{e^{i\omega}-\beta}{1+\beta e^{i\omega}}\right)^{n-m} \left(\frac{e^{i\omega}+\beta}{1-\beta e^{i\omega}}\right)^m \frac{e^{i\omega}\, d\omega}{(e^{i\omega}-\beta)\,\Omega(e^{-i\omega})}$$

$$= i^{n-2m-1} \int_{\mathbb{T}} \left(\frac{\zeta-\beta}{1+\beta\zeta}\right)^{n-m} \left(\frac{\zeta+\beta}{1-\beta\zeta}\right)^m \frac{d\zeta}{(\zeta-\beta)\,\Omega(1/\zeta)} = 0,$$

as $\Omega(z) \neq 0$ in $\overline{\mathbb{C}}\backslash\mathbb{D}$. Under an appropriate choice of the constant $K_n(\Omega)$ the polynomials $\varphi_n$ are orthonormal,

$$\begin{aligned} 1 &= \frac{1}{2\pi} \int_{\alpha}^{2\pi-\alpha} |\varphi_n(e^{i\vartheta};\Omega)|^2 \, W(e^{i\vartheta};\Omega)\, d\vartheta \\ &= \frac{\kappa_n(\Omega)}{2\pi} \int_{\alpha}^{2\pi-\alpha} \varphi_n(e^{i\vartheta};\Omega)\, e^{-in\vartheta} W(e^{i\vartheta};\Omega)\, d\vartheta \\ &= \kappa_n(\Omega)\, K_n \frac{i^{-n}}{2\pi i} \int_{\mathbb{T}} \left(\frac{\zeta+\beta}{1-\beta\zeta}\right)^n \frac{d\zeta}{(\zeta-\beta)\,\Omega(1/\zeta)} \\ &= \kappa_n(\Omega)\, K_n \frac{\gamma^n}{\Omega(1/\beta)} \end{aligned}$$

and the expression for $|K_n|$ drops out immediately from (25) and (26):

$$|K_n|^2 = \frac{2\sin(\alpha/2)}{1+\sin(\alpha/2)}, \qquad n \geqslant n_0(\Omega). \tag{28}$$

Going back to the leading coefficients (25), we see that

$$\kappa_n(\Omega)\,\gamma^n = \left|\Omega\left(\frac{1}{\beta}\right)\right| \sqrt{\frac{1+\sin(\alpha/2)}{2\sin(\alpha/2)}}, \qquad n \geqslant n_0(\Omega). \tag{29}$$

Note that the *reflection coefficients* $a_n(\Omega)$, corresponding to the weight function (27), can be easily computed

$$\begin{gathered} a_n(\Omega) \stackrel{\text{def}}{=} \frac{\varphi_n(0;\Omega)}{\kappa_n(\Omega)} = \sin\frac{\alpha}{2}\, e^{2it}, \\ t = \arg\Omega\left(\frac{1}{\beta}\right), \qquad n \geqslant \max\{n_0(\Omega), 2\}, \end{gathered} \tag{30}$$

that is, the reflection coefficients are constant for large enough $n$.

EXAMPLE 5. For $\Omega(v) = 1$ we have by (22) and (30)

$$a_1(\Omega) = \sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\tan\eta, \qquad a_n(\Omega) = \sin\frac{\alpha}{2}, \qquad n = 2, 3, \ldots.$$

Another interesting example is given by

$$\hat{\Omega}(v) \stackrel{\text{def}}{=} \frac{v^2-\beta^2}{v^2-1},$$

wherein $a_n(\hat{\Omega}) = \sin(\alpha/2)$, $n = 1, 2, \ldots$ . As

$$|\hat{\Omega}(e^{i\omega})|^{-2} = \left|\frac{e^{2i\omega}-1}{e^{2i\omega}-\beta^2}\right|^2 = \frac{4(1+\sin(\alpha/2))^2}{\sin^2\alpha}\left(\cos^2\frac{\alpha}{2} - \cos^2\frac{\vartheta}{2}\right),$$

the weight function $W$ in (27) may be recognized as the weight function, corresponding to Geronimus polynomials with positive reflection coefficients (cf. [5, formula (XI.26), p. 94]):

$$W(e^{i\vartheta}; \hat{\Omega}) = 2\left(1+\sin\frac{\alpha}{2}\right)^2 \frac{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}{\sin(\vartheta/2)}.$$

It seems relevant to evaluate now the reversed $*$-polynomials $\varphi_n^*$. By using the second symmetry relation (7) for $h$ and the similar one for $w$, we obtain

$$\varphi_n^*(z; \Omega) = z^n \overline{\varphi_n(1/\bar{z}; \Omega)} = \overline{K_n}\left\{\frac{\Omega(1/v)}{1+\beta v}\, w^n(v) + \frac{v\Omega(v)}{v+\beta}\, w^n\left(\frac{1}{v}\right)\right\}, \qquad (31)$$

that very much resembles (22).

From this point on, we refer to the orthonormal polynomials $\varphi_n$ in (22) as *Akhiezer's polynomials for the arc* $\Delta_\alpha$.

*Outer Functions for the Arc* $\Delta_\alpha$. Let $\rho(e^{i\vartheta})$ be a nonnegative measurable function which satisfies the Szegő condition for the arc $\Delta_\alpha$

$$\int_\alpha^{2\pi-\alpha} \frac{|\log \rho(e^{i\vartheta})|}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\, d\vartheta < \infty. \qquad (32)$$

By an *outer function for* $\Delta_\alpha$ we mean here a function $g(z; \rho)$ which is analytic and nonvanishing in $\mathbb{C}\backslash\Delta_\alpha$, $g(\infty; \rho) > 0$, and

$$|g(e^{i\vartheta}; \rho)|^{-2} = \rho(e^{i\vartheta}) \qquad \text{a.e. on } \Delta_\alpha.$$

Such a function does exist under the Szegő condition (32) and can be easily found from the well known outer function for the unit circle. Indeed, consider the measurable function $\tilde{\rho}(e^{i\omega}) = \rho(h(e^{i\omega}))$ on the unit circle. By the change of variables formula (19), we see that the Szegő condition (32) is equivalent to the standard Szegő condition for $\tilde{\rho}$ on the unit circle, that is, $\log \tilde{\rho} \in L^1(\mathbb{T})$. We commence with the ordinary outer function

$$\tilde{g}(v; \tilde{\rho}) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\omega}+v}{e^{i\omega}-v} \log\frac{1}{\tilde{\rho}(e^{i\omega})}\, d\omega + i\delta\right\}.$$

By the symmetry property (8) the equality $\tilde{\rho}(e^{i\omega}) = \tilde{\rho}(e^{-i\omega})$ holds and hence

$$\tilde{g}(v; \tilde{\rho}) = \exp\left\{\frac{1}{4\pi}\int_0^{\pi}\left(\frac{1+ve^{-i\omega}}{1-ve^{-i\omega}} + \frac{1+ve^{i\omega}}{1-ve^{i\omega}}\right)\log\frac{1}{\tilde{\rho}(e^{i\omega})}\,d\omega + i\delta\right\}$$
$$= \exp\left\{\frac{1}{2\pi}\int_0^{\pi}\frac{1-v^2}{1+v^2-2v\cos\omega}\log\frac{1}{\tilde{\rho}(e^{i\omega})}\,\omega + i\delta\right\}.$$

A real constant $\delta$ is chosen to meet $\tilde{g}(-\beta; \tilde{\rho}) > 0$. Computing

$$\frac{1-\beta^2}{1+\beta^2+2\beta\cos\omega} = \frac{1}{\sin(\alpha/2) + i\cos(\alpha/2)\cos\omega} = \frac{\sin(\alpha/2) - i\cos(\alpha/2)\cos\omega}{\sin^2(\alpha/2) + \cos^2(\alpha/2)\cos^2\omega}$$

shows that

$$\delta = \frac{1}{2\pi}\int_0^{\pi}\frac{\cos(\alpha/2)\cos\omega}{\sin^2(\alpha/2)+\cos^2(\alpha/2)\cos^2\omega}\log\frac{1}{\tilde{\rho}(e^{i\omega})}\,d\omega.$$

Therefore,

$$\tilde{g}(v; \tilde{\rho}) = \exp\left\{\frac{1}{2\pi}\int_0^{\pi}\left(\frac{1-v^2}{1+v^2-2v\cos\omega} + i\frac{\cos(\alpha/2)\cos\omega}{\sin^2(\alpha/2)+\cos^2(\alpha/2)\cos^2\omega}\right)\right.$$
$$\left.\times\log\frac{1}{\tilde{\rho}(e^{i\omega})}\,d\omega\right\}. \quad (33)$$

It is clear that the function $g(z; \rho) = \tilde{g}(v; \tilde{\rho})$, $z = h(v)$, is an outer function for $\Delta_\alpha$. Thus by (9) and (19) we have

$$g(z; \rho) = \exp\left\{\frac{1}{2\pi}\int_0^{\pi}\frac{1-v^2}{1+v^2-2v\cos\omega}\log\frac{1}{\tilde{\rho}(e^{i\omega})}\,d\omega\right\}$$
$$\times\exp\left\{\frac{i}{4\pi}\int_{\alpha}^{2\pi-\alpha}\log\frac{1}{\rho(e^{i\vartheta})}\frac{\cos(\vartheta/2)\,d\vartheta}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\right\}. \quad (34)$$

The function $\tilde{g} \in H^2$ and thereby admits boundary values a.e. on the unit circle. The same is then true for $g$ and the arc $\Delta_\alpha$. We denote by $g_{\pm}(e^{i\vartheta}; \rho)$ the boundary values of $g$ on $\Delta_\alpha$ from outside and inside of the unit disk, respectively. They can be calculated by invoking the standard method for computing the limit values of Poisson's integral and its conjugate function (cf., e.g., [8, Chap. 1.15] for the unit circle case). In our situation the corresponding formula takes on the form

$$g_{\pm}(e^{i\vartheta_1};\rho)=\frac{1}{\sqrt{\rho(e^{i\vartheta_1})}}\exp\left\{\frac{i}{4\pi}\int_{\alpha}^{2\pi-\alpha}\log\frac{1}{\rho(e^{i\vartheta})}\frac{\cos(\vartheta/2)\,d\vartheta}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\right\}$$
$$\times\exp\left\{\pm\frac{i}{4\pi}\,\text{v.p.}\int_{\alpha}^{2\pi-\alpha}\log\frac{1}{\rho(e^{i\vartheta})}\right.$$
$$\left.\times\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}\right\} \tag{35}$$

for $\vartheta_1\in\Delta_\alpha$. The latter integral converges absolutely whenever $\rho(e^{i\vartheta})$ is positive and satisfies the Lipschitz condition with some positive exponent. We shall return to this relation in more detail later in Section 5.

EXAMPLE 6. For $\Omega\in\mathcal{M}$, let

$$\rho(e^{i\vartheta};\Omega)=\rho_\Omega(e^{i\vartheta})\overset{\text{def}}{=}\frac{\sin(\alpha/2)}{2\sin^2(\vartheta/2)\,|\Omega(e^{i\omega})|^2},\qquad e^{i\vartheta}=h(e^{i\omega})\in\Delta_\alpha. \tag{36}$$

The function $\rho$ obviously satisfies the Szegő condition (32). Direct calculation based on (6) gives

$$\sin^2\frac{\vartheta}{2}=\frac{2-h(e^{i\omega})-h^{-1}(e^{i\omega})}{4}$$
$$=\frac{4\sin^2(\alpha/2)}{(1+\sin(\alpha/2))^2}\frac{1}{(1-\beta^2e^{2i\omega})(1-\beta^2e^{-2i\omega})}.$$

Set

$$g_\Omega(z)\overset{\text{def}}{=}A_\Omega\frac{\Omega(1/v)}{1-\beta^2v^2},\qquad A_\Omega=\frac{2\sqrt{2\sin(\alpha/2)}}{1+\sin(\alpha/2)}e^{iu}, \tag{37}$$

where a real constant $u$ is chosen to meet $g_\Omega(\infty)>0$: $u=\arg\Omega(1/\beta)$. By the assumption on the zeros of $\Omega$ the function $g_\Omega$ is analytic and nonvanishing in $\mathbb{C}\backslash\Delta_\alpha$. For its boundary values the equality

$$|g_\Omega(e^{i\vartheta})|^{-2}=|A_\Omega|^{-2}\frac{(1-\beta^2e^{2i\omega})(1-\beta^2e^{-2i\omega})}{\Omega(e^{i\omega})\,\Omega(e^{-i\omega})}=\rho_\Omega(e^{i\vartheta}) \tag{38}$$

holds, that is, $g_\Omega(z)=g(z;\rho_\Omega)$.

The relation (cf. (27))

$$W(e^{i\vartheta};\Omega)=\rho_\Omega(e^{i\vartheta})\frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}} \tag{39}$$

plays a key role throughout the rest of the paper.

We need hereafter another, distinct from (29), expression for the leading coefficient $\kappa_n(\Omega)$. Put in (37) $v=\beta \Leftrightarrow z=0$,

$$\left|\Omega\left(\frac{1}{\beta}\right)\right| = (1-\beta^4)\,|A_\Omega|^{-1}\,|g(0;\rho_\Omega)|. \tag{40}$$

In view of (33) and (34) the equality

$$|g(0;\rho_\Omega)| = |\tilde{g}(\beta;\widetilde{\rho_\Omega})|\exp\left\{\frac{1}{2\pi}\int_0^\pi \Re\left(\frac{1-\beta^2}{1+v^2-2\beta\cos\omega}\right)\log\frac{1}{\widetilde{\rho_\Omega}(e^{i\omega})}\,d\omega\right\}$$

is valid. Since

$$\Re\left(\frac{1-\beta^2}{1+v^2-2\beta\cos\omega}\right) = \frac{\sin(\alpha/2)}{\sin^2(\alpha/2)+\cos^2(\alpha/2)\cos^2\omega} = \frac{\sin^2(\vartheta/2)}{\sin(\alpha/2)},$$

we see by the change of variables formula (19) that

$$|g(0;\rho_\Omega)| = \exp\left\{\frac{1}{4\pi}\int_\alpha^{2\pi-\alpha}\frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\log\frac{1}{\rho_\Omega(e^{i\vartheta})}\,d\vartheta\right\}.$$

The desirable expression emerges now from (29) and (40): for $n \geqslant n_0(\Omega)$

$$\kappa_n(\Omega)\,\gamma^n = \frac{1}{\sqrt{1+\sin(\alpha/2)}}\exp\left\{\frac{1}{4\pi}\int_\alpha^{2\pi-\alpha}\frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}} \times\log\frac{1}{\rho_\Omega(e^{i\vartheta})}\,d\vartheta\right\}. \tag{41}$$

## 3. ASYMPTOTIC RELATIONS FOR AKHIEZER'S POLYNOMIALS

*Asymptotics Off the Arc $\Delta_\alpha$.* An explicit expression (22) for Akhiezer's polynomials makes it possible studying their asymptotic behavior.

Let $z \in \mathbb{C}\setminus\Delta_\alpha \Leftrightarrow |v|<1$. Then

$$\varphi_n(z;\Omega) = \Lambda_n(v;\Omega) + \Lambda_n\left(\frac{1}{v};\Omega\right), \tag{42}$$

where in view of Example 6

$$\Lambda_n(v;\Omega) = K_n\frac{\Omega(1/v)}{1-\beta v}\,w^n(v)$$
$$= K_n A_\Omega^{-1}(1+\beta v)\,g(z;\rho_\Omega)\,w^n(v).$$

It remains to express this value through the variable $z$. As far as $w(v)$ goes, it is displayed in (15). By (26), (28), and (37) we have

$$K_n A_\Omega^{-1} = \frac{\sqrt{1+\sin(\alpha/2)}}{2}. \tag{43}$$

To express $v$ in terms of $z$, notice that the simple manipulation with (6) and (12) provides

$$v(z-1) = -2v^2 \frac{1+\beta^2}{(v+\beta)(1+\beta v)} = \frac{2}{i} w(v) - \frac{z+1}{\beta},$$

so that

$$v = \frac{2}{i}\frac{w(v)}{z-1} + i\frac{1+\sin(\alpha/2)}{\cos(\alpha/2)}\frac{z+1}{z-1}.$$

Hence (cf. (15))

$$v = \frac{\sqrt{(z+1)^2-4\gamma^2 z} - (z+1)\sin(\alpha/2)}{i\gamma(z-1)} = \frac{R(z)-(z+1)\sin(\alpha/2)}{i\gamma(z-1)} \tag{44}$$

and

$$1+\beta v = \frac{1}{1+\sin(\alpha/2)} \frac{z-1-2\sin(\alpha/2)+R(z)}{z-1}. \tag{45}$$

If the function $\Omega$ is fixed, i.e., it does not depend on $n$, the second term in the right hand side of (42) decays exponentially uniformly on compact sets inside $\mathbb{C}\backslash\Delta_\alpha$, that leads to the following.

Proposition 2. *Given $\Omega \in \mathscr{M}$ the asymptotic relation*

$$\begin{aligned} \varphi_n(z;\Omega) &= \frac{z-1-2\sin(\alpha/2)+\sqrt{(z+1)^2-4\gamma^2 z}}{2\sqrt{1+\sin(\alpha/2)}(z-1)}\, g(z;\rho_\Omega)\, w^n(v) + \varepsilon_n(z), \\ w(v) &= \frac{z+1+\sqrt{(z+1)^2-4\gamma^2 z}}{2\gamma} \end{aligned} \tag{46}$$

*holds, where $\varepsilon_n$ decays exponentially uniformly on compact sets inside $\mathbb{C}\backslash\Delta_\alpha$ as $n \to \infty$.*

*Remark* 3. Keeping in mind further considerations, we should stress that in the sequel the function $\Omega$ *does depend* on $n$, so we should be much more accurate while evaluating the second term in (42).

*Asymptotics on the Arc $\Delta_\alpha$.* The passage to the limit in (22), when $v$ goes to the unit circle, needs to be clarified. Let $v \to e^{i\omega}$, $0 < \omega < \pi$. As it was shown at the beginning of Section 2, now $z \to e^{i\vartheta} \in \Delta_\alpha^-$ and $g(z; \rho_\Omega) \to g_-(e^{i\vartheta}; \rho_\Omega)$, which is the interior boundary value for the outer function $g(z; \rho_\Omega)$. Hence,

$$\Lambda_n(e^{i\omega}) = K_n A_\Omega^{-1}(1 + \beta e^{i\omega})\, g_-(e^{i\vartheta}; \rho_\Omega)\, w^n(e^{i\omega}).$$

By (43), (45), and (17), taking an appropriate sign for the square root (see discussion before the formula (16)), we have

$$\begin{aligned}
K_n A_\Omega^{-1}(1 + \beta e^{i\omega}) &= \frac{e^{i\vartheta} - 1 - 2\sin(\alpha/2) - 2ie^{i(\vartheta/2)}\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}{2\sqrt{1 + \sin(\alpha/2)}\,(e^{i\vartheta} - 1)} \\
&= \frac{i\sin(\vartheta/2) - \sin(\alpha/2)\, e^{i(\vartheta/2)} - i\sin\lambda\cos(\alpha/2)}{2i\sin(\vartheta/2)\sqrt{1 + \sin(\alpha/2)}} \\
&= \frac{\cos(\alpha/2)\, e^{-i\lambda} - (1 + \sin(\alpha/2))\, e^{-i(\vartheta/2)}}{2i\sin(\vartheta/2)\sqrt{1 + \sin(\alpha/2)}}.
\end{aligned}$$

Applying (18), we get

$$\begin{aligned}
\Lambda_n(e^{i\omega}) &= \frac{\cos(\alpha/2)\, e^{-i\lambda} - (1 + \sin(\alpha/2))\, e^{-i(\vartheta/2)}}{2i\sin(\vartheta/2)\sqrt{1 + \sin(\alpha/2)}} \\
&\quad \times \exp\left\{in\left(\frac{\vartheta}{2} - \lambda\right)\right\} g_-(e^{i\vartheta}; \rho_\Omega).
\end{aligned}$$

In a similar fashion, we obtain the expression for $\Lambda_n(e^{-i\omega})$.

Proposition 4. *For $e^{i\vartheta} \in \Delta_\alpha$ the equality*

$$\begin{aligned}
\varphi_n(e^{i\vartheta}; \Omega) &= \frac{e^{-i\lambda}\sqrt{1 - \sin(\alpha/2)} - e^{-i(\vartheta/2)}\sqrt{1 + \sin(\alpha/2)}}{2i\sin(\vartheta/2)} \\
&\quad \times \exp\left\{in\left(\frac{\vartheta}{2} - \lambda\right)\right\} g_-(e^{i\vartheta}; \rho_\Omega) \\
&\quad + \frac{e^{i\lambda}\sqrt{1 - \sin(\alpha/2)} - e^{-i(\vartheta/2)}\sqrt{1 + \sin(\alpha/2)}}{2i\sin(\vartheta/2)} \\
&\quad \times \exp\left\{in\left(\frac{\vartheta}{2} + \lambda\right)\right\} g_+(e^{i\vartheta}; \rho_\Omega)
\end{aligned} \tag{47}$$

*is valid.*

## 4. BERNSTEIN–SZEGŐ METHOD FOR A CIRCULAR ARC

*Approximation by Special Weight Functions.* The approximation of an arbitrary weight function $W$ by special weight functions $W_q$, which makes it possible studying the asymptotic behavior of orthonormal polynomials $\varphi_n(z; W)$ by comparing them to orthonormal polynomials $\varphi_n(z; W_q)$, underlies the Bernstein–Szegő method. In the case of a circular arc such special weight functions take on the form (27) with $\Omega \in \mathscr{M}$. The asymptotic behavior for $\varphi_n(z; W_q)$ was exhibited in Section 3.

Given an arbitrary weight function $W$ on the arc $\Delta_\alpha$, let

$$\rho(e^{i\vartheta}) \overset{\text{def}}{=} W(e^{i\vartheta}) \frac{\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}{\sin(\vartheta/2)} \tag{48}$$

(cf. (39)). We assume in what follows that

$$\rho \in C(\Delta_\alpha), \qquad 0 < l \leqslant \rho(e^{i\vartheta}) \leqslant L < \infty. \tag{49}$$

By the *modulus of continuity* of a function $p$ on $\Delta_\alpha$ we always mean the function

$$\omega(x, p) \overset{\text{def}}{=} \max_{|\vartheta_1 - \vartheta_2| \leqslant x} |p(e^{i\vartheta_1}) - p(e^{i\vartheta_2})|,$$

where the maximum is taken over all pairs $(\vartheta_1, \vartheta_2)$ from the interval $[\alpha, 2\pi - \alpha]$. The same notation is kept for continuous functions on the whole unit circle.

Along with the function $\rho$, consider an auxiliary function

$$f(e^{i\vartheta}) \overset{\text{def}}{=} \frac{\sin(\alpha/2)}{2\sin^2(\vartheta/2)\,\rho(e^{i\vartheta})} = \frac{\sin(\alpha/2)}{2\sin(\vartheta/2)\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}\,W(e^{i\vartheta})}, \tag{50}$$

which is positive and continuous on $\Delta_\alpha$ by (49). Next, it is convenient to go over from the arc $\Delta_\alpha$ to the unit circle (cf. (6) and (11)):

$$\begin{aligned} e^{i\vartheta} &= h(e^{i\omega}), \qquad & f(e^{i\vartheta}) &= F(e^{i\omega}), \\ e^{i\omega} &= v(e^{i\xi}), \qquad & F(e^{i\omega}) &= G(e^{i\xi}). \end{aligned} \tag{51}$$

Due to the property (10) of $h'(v)$ such transitions do not alter moduli of continuity essentially (in the sense of order):

$$\omega(x, G) \leqslant C_0\omega(x, F) \leqslant C_1\omega(x, f) \leqslant C_2\omega(x, \rho). \tag{52}$$

Throughout the rest of the paper $C_k$, $k=0, 1, ...,$ stand for positive constants, which depend only on $\alpha$ and given weight function $W$. It is convenient to note here that by the symmetry property (8) we have

$$F(e^{i\omega}) = F(e^{-i\omega}), \qquad G(e^{i\xi}) = G(e^{i\xi_-}), \tag{53}$$

where $v(e^{i\xi_-}) = e^{-i\omega}$.

Given a positive integer $n \geqslant 4$, set

$$q \stackrel{\text{def}}{=} \left[\frac{n}{2}\right] - 1. \tag{54}$$

We can approximate a positive and continuous $2\pi$-periodic function $\tilde{G}(\xi) \stackrel{\text{def}}{=} \sqrt{G(e^{i\xi})}$ uniformly by positive trigonometric polynomials (e.g., by Jackson's polynomials)

$$0 < S_q(\xi) = \sum_{k=-q}^{q} A_k e^{ik\xi}, \qquad A_{-k} = \overline{A_k},$$

that is,

$$\|\tilde{G}(\xi) - S_q(\xi)\|_\infty \leqslant 12\omega\left(\frac{1}{q+1}, \tilde{G}\right) \leqslant C_3\omega\left(\frac{1}{n}, \rho\right). \tag{55}$$

Here $\|\cdot\|_\infty$ denotes the uniform norm on the unit circle or on the arc $\Delta_\alpha$ (that is always clear from the context). Note that with no loss of generality we may presume

$$\tilde{G}(\xi) \leqslant S_q(\xi) \leqslant C_4. \tag{56}$$

It follows from (51) that

$$\begin{aligned} 0 < S_q(\xi) &= \sum_{k=-q}^{q} A_k \left(i\,\frac{1-\beta e^{i\omega}}{e^{i\omega}+\beta}\right)^k \\ &= \frac{P_{2q}(e^{i\omega})}{(e^{i\omega}+\beta)^q\,(1-\beta e^{i\omega})^q} = \frac{P_{2q}(e^{i\omega})}{e^{iq\omega}\,|1-\beta e^{i\omega}|^{2q}}. \end{aligned} \tag{57}$$

By F. Riesz's theorem the positive trigonometric polynomial $P_{2q}(e^{i\omega})\,e^{-iq\omega}$ in (57) admits a representation of the form

$$P_{2q}(e^{i\omega})\,e^{-iq\omega} = B_q^2 \prod_{v=1}^{q} |e^{i\omega} - c_v^{(q)}|^2, \qquad |c_v^{(q)}| < 1, \qquad B_q > 0 \tag{58}$$

(some of the $c_\nu^{(q)}$ may be zeros). Put

$$H_q(v) = B_q(v+\beta)^{-q} \prod_{\nu=1}^{q} (v - c_\nu^{(q)}), \qquad v \in \mathbb{C}.$$

Then $S_q(\xi) = |H_q(e^{i\omega})|^2$ and the function $\Omega_q(v) \stackrel{\text{def}}{=} H_q(v)\, \bar{H}_q(v)$ satisfies

$$\Omega_q(v) = \frac{B_q^2}{(v^2-\beta^2)^q} \prod_{\nu=1}^{q} (v - c_\nu^{(q)})(v - \overline{c_\nu^{(q)}}) = \frac{T_{2q}(v)}{(v^2-\beta^2)^q}, \tag{59}$$

where as usual $\bar{H}_q(v) = \overline{H_q(\bar{v})}$. We see that $\Omega_q$ is of the form (24) (cf. Example 4 in Section 2). Besides, $\Omega_q(x) \geqslant 0$ for real $x$ and all zeros of $\Omega_q$ lie inside $\mathbb{D}$, so that $\Omega_q \in \mathscr{M}$. Next

$$|\Omega_q(e^{i\omega})|^2 = \Omega_q(e^{i\omega})\, \Omega_q(e^{-i\omega}) = |H_q(e^{i\omega})\, H_q(e^{-i\omega})|^2 = S_q^2(\xi). \tag{60}$$

We are now in a position to turn to the approximation of the functions $f$ and $\rho$. By (55), (53), and (56), we get

$$\| f(e^{i\vartheta}) - |\Omega_q(e^{i\omega})|^2 \|_\infty \leqslant C_5 \omega\left(\frac{1}{n}, \rho\right) \tag{61}$$

and

$$f(e^{i\vartheta}) \leqslant |\Omega_q(e^{i\omega})|^2 \leqslant C_4^2. \tag{62}$$

The latter inequality enables one to evaluate the reciprocal values. Indeed,

$$\left| \frac{1}{f(e^{i\vartheta})} - \frac{1}{|\Omega_q(e^{i\omega})|^2} \right| \leqslant \frac{|f(e^{i\vartheta}) - |\Omega_q(e^{i\omega})|^2|}{|f(e^{i\vartheta})|^2},$$

and as the moduli of continuity of $f$ and $\rho$ have the same order, we come to the conclusion

$$\| \rho(e^{i\vartheta}) - \rho(e^{i\vartheta}; \Omega_q) \|_\infty \leqslant C_6 \omega\left(\frac{1}{n}, \rho\right), \tag{63}$$

where notation

$$\rho(e^{i\vartheta}; \Omega_q) = \rho_q(e^{i\vartheta}) \stackrel{\text{def}}{=} \frac{\sin(\alpha/2)}{2\sin^2(\vartheta/2)\, |\Omega_q(e^{i\omega})|^2} \tag{64}$$

is consistent with (36). The inequality $\rho(e^{i\vartheta}; \Omega_q) \leqslant \rho(e^{i\vartheta})$, which proves useful later on, is a simple consequence of (62) and (50).

We can modify the inequality (63) if we set

$$W(e^{i\vartheta};\Omega_q)=W_q(e^{i\vartheta})\stackrel{\text{def}}{=}\rho_q(e^{i\vartheta})\frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}} \tag{65}$$

(cf. (39)). Namely

$$\left\|1-\frac{\rho_q(e^{i\vartheta})}{\rho(e^{i\vartheta})}\right\|_\infty=\left\|1-\frac{W_q(e^{i\vartheta})}{W(e^{i\vartheta})}\right\|_\infty\leqslant C_7\omega\left(\frac{1}{n},\rho\right). \tag{66}$$

Due to (62) the similar relation holds for the reciprocal values

$$\left\|1-\frac{\rho(e^{i\vartheta})}{\rho_q(e^{i\vartheta})}\right\|_\infty=\left\|1-\frac{W(e^{i\vartheta})}{W_q(e^{i\vartheta})}\right\|_\infty\leqslant C_8\omega\left(\frac{1}{n},\rho\right). \tag{67}$$

*Remark* 5. The above consideration shows that

$$\lim_{n\to\infty}\|\rho_q^{\pm 1}(e^{i\vartheta})-\rho^{\pm 1}(e^{i\vartheta})\|_\infty=0$$

as long as $\lim_{n\to\infty} q(n)=\infty$ (no rate of convergence can be claimed in general).

*Bernstein–Korous Identity*. Let $\{\varphi_{n,j}(z,\mu_j)=\kappa_{n,j}z^n+\cdots\}_0^\infty$ be orthonormal polynomials systems with respect to measures $\mu_j$, $j=1,2$. We recall the identity, connecting polynomials $\varphi_{n,1}$ and $\varphi_{n,2}$. Expansion of the polynomial $\varphi_{n,1}$ over the system $\{\varphi_{k,2}\}_0^n$ gives

$$\varphi_{n,1}(z)=\sum_{k=0}^n d_{k,n}\varphi_{k,2}(z),\qquad d_{k,n}=\frac{1}{2\pi}\int_0^{2\pi}\varphi_{n,1}(e^{i\vartheta})\,\overline{\varphi_{k,2}(e^{i\vartheta})}\,d\mu_2,$$

whence it follows that

$$\begin{aligned}\varphi_{n,1}(z)&=\frac{1}{2\pi}\int_0^{2\pi}\varphi_{n,1}(e^{i\vartheta})\sum_{k=0}^n\varphi_{k,2}(z)\,\overline{\varphi_{k,2}(e^{i\vartheta})}\,d\mu_2\\&=\frac{1}{2\pi}\int_0^{2\pi}K_{n+1,2}(z,e^{i\vartheta})\,\varphi_{n,1}(e^{i\vartheta})\,d\mu_2\\&=\frac{1}{2\pi}\int_0^{2\pi}K_{n,2}(z,e^{i\vartheta})\,\varphi_{n,1}(e^{i\vartheta})\,d\mu_2\\&\quad+\frac{1}{2\pi}\int_0^{2\pi}\varphi_{n,2}(z)\,\varphi_{n,1}(e^{i\vartheta})\,\overline{\varphi_{n,2}(e^{i\vartheta})}\,d\mu_2.\end{aligned}$$

Here

$$K_{m,2}(u,v) \overset{\text{def}}{=} \sum_{k=0}^{m-1} \varphi_{k,2}(u)\,\overline{\varphi_{k,2}(v)}$$
$$= \frac{\varphi^*_{m,2}(u)\,\overline{\varphi^*_{m,2}(v)} - \varphi_{m,2}(u)\,\overline{\varphi_{m,2}(v)}}{1-u\bar{v}} \tag{68}$$

(the latter is known as the Christoffel–Darboux formula, see [8, p. 41, formula (1)]). By the orthogonality property, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_{n,1}(e^{i\vartheta})\,\overline{\varphi_{n,2}(e^{i\vartheta})}\,d\mu_2 = \frac{\kappa_{n,1}}{2\pi}\int_0^{2\pi} e^{in\vartheta}\,\overline{\varphi_{n,2}(e^{i\vartheta})}\,d\mu_2 = \frac{\kappa_{n,1}}{\kappa_{n,2}},$$

so that

$$\varphi_{n,1}(z) = \frac{1}{2\pi}\int_0^{2\pi} K_{n,2}(z,e^{i\vartheta})\,\varphi_{n,1}(e^{i\vartheta})\,d\mu_2 + \frac{\kappa_{n,1}}{\kappa_{n,2}}\,\varphi_{n,2}(z). \tag{69}$$

This equality can be rewritten in terms of monic polynomials as

$$\Phi_{n,1}(z) - \Phi_{n,2}(z) = \frac{1}{2\pi}\int_0^{2\pi} K_{n,2}(z,e^{i\vartheta})\,\Phi_{n,1}(e^{i\vartheta})\,d\mu_2$$
$$= \frac{1}{2\pi}\int_0^{2\pi} K_{n,2}(z,e^{i\vartheta})\,\Phi_{n,1}(e^{i\vartheta})(d\mu_2 - d\mu_1). \tag{70}$$

We shall handle formula (70) in the following situation:

$\Phi_{n,1}(z) = \Phi_n(z)$—monic orthogonal polynomials with respect to the weight function $W(e^{i\vartheta})$ on $\Delta_\alpha$;

$\Phi_{n,2}(z) = \Phi_n(z;\Omega_q)$—monic orthogonal polynomials with respect to the weight function $W_q(e^{i\vartheta})$ (65).

Finally, we arrive at the relation, which is referred to as the *Bernstein–Korous identity*

$$\Phi_n(z) - \Phi_n(z;\Omega_q)$$
$$= \frac{1}{2\pi}\int_\alpha^{2\pi-\alpha} K_n(z,e^{i\vartheta};\Omega_q)\,\Phi_n(e^{i\vartheta})\,W(e^{i\vartheta})\left(\frac{W_q(e^{i\vartheta})}{W(e^{i\vartheta})} - 1\right)d\vartheta. \tag{71}$$

## 5. ASYMPTOTIC RELATIONS FOR GENERAL ORTHOGONAL POLYNOMIALS

*Asymptotics for the Leading Coefficient.* We begin with the weight function $W$ on $\Delta_\alpha$, which satisfies (48), (49), and the corresponding sequence of monic orthogonal polynomials $\Phi_n = \Phi_n(W)$. According to the well known extremal property of orthogonal polynomials the relation

$$\kappa_n^{-2}(W) = \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} |\Phi_n(e^{i\vartheta}, W)|^2 \, W(e^{i\vartheta}) \, d\vartheta$$
$$= \min \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} |P(e^{i\vartheta})|^2 \, W(e^{i\vartheta}) \, d\vartheta$$

holds for the leading coefficient $\kappa_n = \kappa_n(W)$, where the minimum is taken over all monic polynomials of degree $n$. Therefore the asymptotic behavior of $\kappa_n$ is of particular interest. The Bernstein–Szegő method developed in Section 4 provides a technique for studying this problem.

Denote

$$\|f\|_W^2 \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} |f(e^{i\vartheta})|^2 \, W(e^{i\vartheta}) \, d\vartheta.$$

Then, for the approximating sequence of weight functions $W_q$ (65) and monic orthogonal polynomials $\Phi_n(z; \Omega_q)$, we have

$$\|\Phi_n\|_W^2 \leqslant \|\Phi_n(\Omega_n)\|_W^2$$
$$= \|\Phi_n(\Omega_q)\|_{W_q}^2 + \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} |\Phi_n(e^{i\vartheta}, \Omega_q)|^2 \, W_q(e^{i\vartheta}) \left( \frac{W(e^{i\vartheta})}{W_q(e^{i\vartheta})} - 1 \right) d\vartheta.$$

It follows now from (67) that

$$\|\Phi_n\|_W^2 \leqslant \|\Phi_n(\Omega_q)\|_{W_q}^2 + \left\| 1 - \frac{W(e^{i\vartheta})}{W_q(e^{i\vartheta})} \right\|_\infty \|\Phi_n(\Omega_q)\|_{W_q}^2$$
$$= \|\Phi_n(\Omega_q)\|_{W_q}^2 \left( 1 + C_8 \omega\left(\frac{1}{n}, \rho\right)\right).$$

In exactly the same way, we get by (66)

$$\|\Phi_n(\Omega_q)\|_{W_q}^2 \leqslant \|\Phi_n(\Omega_q)\|_W^2 \left( 1 + C_7 \omega\left(\frac{1}{n}, \rho\right)\right).$$

Thus, we come to the relation

$$\left|\frac{\kappa_n}{\kappa_n(\Omega_q)}-1\right| \leqslant C_9 \omega\left(\frac{1}{n}, \rho\right), \tag{72}$$

which in turn yields $\lim_{n\to\infty} \kappa_n/\kappa_n(\Omega_q)=1$.

Recall now that explicit expression (41) is known for the leading coefficients $\kappa_n(\Omega_q)$ for $n \geqslant q+1$ (the latter inequality holds thanks to the appropriate choice of $q$ in (54)). The final result drops out immediately upon taking $n\to\infty$

$$\lim_{n\to\infty} \kappa_n \gamma^n = \frac{1}{\sqrt{1+\sin(\alpha/2)}} \times \exp\left\{\frac{1}{4\pi}\int_\alpha^{2\pi-\alpha} \frac{\sin(\vartheta/2)}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}} \log\frac{1}{\rho(e^{i\vartheta})}\, d\vartheta\right\}. \tag{73}$$

Note that the asymptotic behavior of the leading coefficients in a much more general setting was investigated in [16, Theorem 12.3; 9, Theorem 1].

*Asymptotics Off the Arc $\Delta_\alpha$.* Let us first make sure that all zeros of the polynomials $\varphi_n$ are being attracted to the arc $\Delta_\alpha$. The latter means that given an arbitrary compact set $K \in \overline{\mathbb{C}} \setminus \Delta_\alpha$, the polynomials $\varphi_n(z; \Omega_q) \neq 0$, $z \in K$ for $n \geqslant n_0(K)$ (cf. [2, Remark before Lemma 4]). Indeed, denote

$$U_q(v) \overset{\text{def}}{=} \frac{v\Omega_q(v)}{\Omega_q(1/v)}\left(\frac{w(1/v)}{w(v)}\right)^{n-1}.$$

By (22) we have

$$\varphi_n(z; \Omega_q) = C_n \frac{\Omega_q(1/v)}{1-\beta v} w^n(v)\left\{1+\frac{v+\beta}{1+\beta v} U_q(v)\right\}. \tag{74}$$

The function $U_q$ is easily estimated with the help of (59) and (12)

$$\frac{\Omega_q(v)}{\Omega_q(1/v)} = \left(\frac{1-\beta^2 v^2}{v^2-\beta^2}\right)^q \prod_{\nu=1}^{q} \frac{(v-c_\nu^{(q)})(v-\overline{c_\nu^{(q)}})}{(1-\overline{c_\nu^{(q)}}v)(1-c_\nu^{(q)}v)},$$

whence it follows that

$$|U_q(v)| \leqslant \left|\frac{1-\beta^2 v^2}{v^2-\beta^2}\right|^q \left|\frac{w(1/v)}{w(v)}\right|^{n-1} = \left|\frac{w(1/v)}{w(v)}\right|^{n-q-1}. \tag{75}$$

Let $\hat{K}$ be the compact set inside $\mathbb{D}$ such that $K=h(\hat{K})$. It is clear that

$$\left|\frac{w(1/v)}{w(v)}\right| \leqslant \delta(k) < 1, \qquad v \in \hat{K},$$

and hence

$$|U_q(v)| \leqslant \delta^{n-q-1}(K), \qquad v \in \hat{K}. \tag{76}$$

The choice of $q$ (54) implies exponential decay of $U_q$ uniformly inside $\mathbb{D}$. The desired property of zeros of $\varphi_n(z; \Omega_q)$ now stems from (74) and the fact that $\Omega_q \neq 0$ outside $\mathbb{D}$.

Our further consideration depends heavily on the Bernstein–Korous identity (71), which can be paraphrased in terms of orthonormal polynomials as

$$\varphi_n(z) - \frac{\kappa_n}{\kappa_n(\Omega_q)}\, \varphi_n(z; \Omega_q) = \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} K_n(z, e^{i\vartheta}; \Omega_q)\, \varphi_n(e^{i\vartheta})(W_q(e^{i\vartheta}) - W(e^{i\vartheta}))\, d\vartheta. \tag{77}$$

Dividing through by $\varphi_n(z; \Omega_q)$ and invoking the Christoffel–Darboux formula (68), we obtain

$$\frac{\varphi_n(z)}{\varphi_n(z; \Omega_q)} - \frac{\kappa_n}{\kappa_n(\Omega_q)} = \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} \left( \frac{\varphi_n^*(z; \Omega_q)}{\varphi_n(z; \Omega_q)}\, \overline{\varphi_n^*(e^{i\vartheta}; \Omega_q)} - \varphi_n(e^{i\vartheta}; \Omega_q) \right) \times \frac{\varphi_n(e^{i\vartheta})\, W(e^{i\vartheta})}{1 - ze^{-i\vartheta}} \left( \frac{W_q(e^{i\vartheta})}{W(e^{i\vartheta})} - 1 \right) d\vartheta. \tag{78}$$

Our goal here is to study the asymptotic behavior of orthogonal polynomials $\varphi_n$ on $K$. To this end note that by the Schwarz inequality, applied to (78), and in view of (66), we have for $z \in K$

$$\left| \frac{\varphi_n(z)}{\varphi_n(z; \Omega_q)} - \frac{\kappa_n}{\kappa_n(\Omega_q)} \right|^2 \leqslant C_{10}(K)\, \omega\left(\frac{1}{n}, \rho\right) \times \int_\alpha^{2\pi-\alpha} \left| \frac{\varphi_n^*(z; \Omega_q)}{\varphi_n(z; \Omega_q)}\, \overline{\varphi_n^*(e^{i\vartheta}; \Omega_q)} - \varphi_n(e^{i\vartheta}; \Omega_g) \right|^2 W(e^{i\vartheta})\, d\vartheta \tag{79}$$

(one has to keep in mind that the $\varphi_n$ are orthonormal with respect to $W$). The ratio in the right hand side of (79) is easily taken care of due to the explicit expressions for Akhiezer's polynomials and their $*$-reversed, so formula (31) comes into play now. In fact,

$$\left|\frac{\varphi_n^*(z;\Omega_q)}{\varphi_n(z;\Omega_q)}\right| = \left|\frac{1+((v-\beta)/(1-\beta v))\,U_q(v)}{1+((v+\beta v)/(1+\beta v))\,U_q(v)}\right|\left|\frac{1-\beta v}{1+\beta v}\right|,$$

and by (76)

$$\frac{\varphi_n^*(z;\Omega_q)}{\varphi_n(z;\Omega_q)} = O(1), \qquad n\to\infty \tag{80}$$

uniformly on $K$.

Next, it is not hard to show that the sequence $\varphi_n(z;\Omega_q)$ is uniformly bounded on the arc $\Delta_\alpha$. Indeed, it is immediate from (22), (28), and (62) that

$$|\varphi_n(e^{i\vartheta};\Omega_q)| \leqslant |K_n(\Omega_q)|\;|\Omega_q(e^{i\omega})|\;(|1-\beta e^{i\omega}|^{-1}+|e^{i\omega}-\beta|^{-1}) \leqslant C_{11}.$$

Finally, taking into account (72), we come to the relation

$$\left|\frac{\varphi_n(z)}{\varphi_n(z;\Omega_q)}-1\right| \leqslant C_{12}(K)\,\omega\left(\frac{1}{n},\rho\right), \qquad z\in K. \tag{81}$$

We are now within easy reach of establishing the asymptotic formula for the orthonormal polynomials $\varphi_n$. Although we are no longer at liberty to apply Proposition 2 directly (cf. Remark 3), due to the relations (74) and (76) the polynomials $\varphi_n(z;\Omega_q)$ behave now exactly as in Section 3. It remains only to note that by (63) and (34) the relation

$$\lim_{n\to\infty}\frac{g(z;\rho_q)}{g(z;\rho)} = 1$$

holds uniformly on $K$, where

$$g(z;\rho) \overset{\text{def}}{=} \exp\left\{\frac{1}{2\pi}\int_0^\pi \frac{1-v^2}{1+v^2-2v\cos\omega}\log\frac{1}{\rho(h(e^{i\omega}))}\,d\omega\right\}$$
$$\times\exp\left\{\frac{i}{4\pi}\int_\alpha^{2\pi-\alpha}\log\frac{1}{\rho(e^{i\vartheta})}\;\frac{\cos(\vartheta/2)\,d\vartheta}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\right\}. \tag{82}$$

Eventually, we reach the following conclusion, which may be recognized as a circular arc analogue of the fundamental Szegő asymptotic formula (2) and which turns into (2) for $\alpha=0$.

THEOREM 6. *Let an arbitrary weight function $W$ on the arc $\Delta_\alpha$ satisfy* (48) *and* (49). *Then, for the orthonormal with respect to $W$ polynomials $\varphi_n$ the asymptotic formula*

$$\varphi_n(z) = \frac{z-1-2\sin(\alpha/2)+\sqrt{(z+1)^2-4\gamma^2 z}}{2\sqrt{1+\sin(\alpha/2)}\,(z-1)}\, g(z;\rho)\, w^n(z)(1+o(1)),$$
$$w(z) = \frac{z+1+\sqrt{(z+1)^2-4\gamma^2 z}}{2\gamma} \tag{83}$$

*holds uniformly on compact subsets of* $\mathbb{C}\backslash\Delta_\alpha$.

The following result is a straightforward consequence of Theorem 6, which is yet worth mentioning.

COROLLARY 7. *For weight functions $W$ under consideration the relative asymptotic formula*

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = \frac{z+1+\sqrt{(z+1)^2-4\gamma^2 z}}{2\gamma}$$

*holds uniformly inside* $\mathbb{C}\backslash\Delta_\alpha$.

Under the much more general Rahmanov's condition such a limit relation was established in [2, Theorem 1].

Theorem 6 provides the asymptotic formula for the reflection coefficients $a_n(W)$.

THEOREM 8. *The reflection coefficients $a_n(W)$, which correspond to weight function $W$* (48), (49) *satisfy*

$$\lim_{n\to\infty} a_n(W) = \sin\frac{\alpha}{2}\, e^{i\tau},$$
$$\tau = \frac{1}{2\pi}\int_\alpha^{2\pi-\alpha} \log\frac{1}{\rho(e^{i\vartheta})}\, \frac{\cos(\vartheta/2)\, d\vartheta}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}.$$

*Proof.* It is easily follows from (81) with $z=0$ and (72) that

$$\lim_{n\to\infty} \frac{a_n(W)}{a_n(\Omega_q)} = 1, \qquad a_n(\Omega_q) = \sin\frac{\alpha}{2}\, e^{2it_q}, \qquad t_q = \arg\Omega_q\left(\frac{1}{\beta}\right) \tag{84}$$

(cf. (30)). To handle the expression for $a_n(\Omega_q)$, we shall go back to Example 6, Section 2, wherein the function (cf. (37))

$$g_q(z) \stackrel{\text{def}}{=} \frac{2\sqrt{2\sin(\alpha/2)}}{1+\sin(\alpha/2)} \frac{\Omega_q(1/v)}{1-\beta^2 v^2} \exp(it_q) \tag{85}$$

is shown to be the outer function with particular limit values (64). From (85) we derive that

$$a_n(\Omega_q) = \sin\frac{\alpha}{2} \frac{g_q(0)}{|g_q(0)|}.$$

The latter quantity can be extracted from the formula for outer functions (34)

$$a_n(\Omega_q) = \sin\frac{\alpha}{2} \exp\left\{\frac{i}{2\pi}\int_\alpha^{2\pi-\alpha} \log\frac{1}{\rho_q(e^{i\vartheta})} \frac{\cos(\vartheta/2)\, d\vartheta}{\sqrt{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\right\}.$$

The statement is now immediate from (84) and inequality (63).[1] ▮

*Remark* 9. Another way of the proving Theorem 8 is a direct computation of the limit

$$\lim_{n\to\infty} a_n(W) = \lim_{n\to\infty} \frac{\phi_n(0)}{\kappa_n},$$

based on the formulas (83), (73) and the change of variables formula (19).

*Remark* 10. The Bernstein–Szegő approximation method provides a rate of convergence in Theorem 8

$$\left| a_n(W) - \sin\frac{\alpha}{2}\, e^{i\tau} \right| \leqslant C\omega\left(\frac{1}{n}, \rho\right).$$

*Asymptotics on the Arc $\Delta_\alpha$.* We begin with the estimate for the derivative $\Omega_q'(v)$ on the unit circle. By (59) and (62) we have

$$\left|\frac{T_{2q}(e^{i\omega})}{(e^{2i\omega}-\beta^2)^q}\right| \leqslant C_4, \qquad |T_{2q}(e^{i\omega})| \leqslant |T(e^{i\omega})|, \qquad T(v) \stackrel{\text{def}}{=} C_4(v^2-\beta^2)^q.$$

[1] The relations (85) and (63) actually imply the uniform convergence of the sequence $\Omega_q(1/v)$ on compact subsets of $\mathbb{D}$.

According to Bernstein's theorem (cf. [3; 14, Sect. 5.1.3, Theorem 1, p. 387])

$$|T'_{2q}(e^{i\omega})| \leqslant |T'(e^{i\omega})| = 2C_4 q \, |e^{2i\omega} - \beta^2|^{q-1}.$$

Next,

$$\begin{aligned} \Omega'_q(v) &= \frac{T'_{2q}(v)}{(v^2-\beta^2)^q} - 2qv \frac{T_{2q}(v)}{(v^2-\beta^2)^{q+1}} \\ &= \frac{T'_{2q}(v)}{(v^2-\beta^2)^q} - \frac{2qv}{v^2-\beta^2} \Omega_q(v), \end{aligned}$$

so that

$$|\Omega'_q(e^{i\omega})| \leqslant 2 \frac{2C_4 q}{|e^{2i\omega}-\beta^2|} \leqslant C_{13} n. \tag{86}$$

From this point on we assume in addition to (49) that the function $\rho$ satisfies *Dini condition*

$$\omega(\rho, x) \log \frac{1}{x} = o(1), \qquad x \to 0. \tag{87}$$

PROPOSITION 11. *For a weight function W, which satisfies the Dini condition* (87), *the orthogonal polynomials* $\varphi_n$ *are uniformly bounded on* $\Delta_\alpha$:

$$M_n \overset{\text{def}}{=} \max_{\Delta_\alpha} |\varphi_n(e^{i\vartheta})| = O(1), \qquad n \to \infty. \tag{88}$$

*Proof.* From the Bernstein–Korous identity (77) with $z = e^{i\vartheta_0} \in \Delta_\alpha$, and (66) we derive

$$\left| \varphi_n(e^{i\vartheta_0}) - \frac{\kappa_n}{\kappa_n(\Omega_q)} \varphi_n(e^{i\vartheta_0}; \Omega_q) \right| \leqslant C_{14} M_n \omega\left(\frac{1}{n}, \rho\right) I_n, \tag{89}$$

where

$$I_n \overset{\text{def}}{=} \int_\alpha^{2\pi-\alpha} |K_n(e^{i\vartheta_0}, e^{i\vartheta}; \Omega_q)| \frac{d\vartheta}{\sqrt{\cos^2(\alpha/2) - \cos^2(\vartheta/2)}}.$$

Somewhat tedious manipulations with the Christoffel kernel $K_n$ based on the Christoffel–Darboux formula and explicit expressions for Akhiezer's polynomials and their reversed (see (22) and (31)) end up with the equality

$$\overline{K_n(e^{i\vartheta_0}, e^{i\vartheta}; \Omega_q)} = \frac{F_1(e^{i\omega})}{e^{i(\omega-\omega_0)}-1} + \frac{F_2(e^{i\omega})}{e^{i\omega}-e^{i\omega_0}},$$
$$F_j(e^{i\omega}) = F_j(e^{i\omega}, e^{i\omega_0}, n), \qquad j = 1, 2,$$

where, as usual, $e^{i\vartheta} = h(e^{i\omega})$, $e^{i\vartheta_0} = h(e^{i\omega_0})$ and

$$\sin\frac{\alpha}{2} F_1(e^{i\omega}) \stackrel{\text{def}}{=} e^{i\omega}\Omega_q(e^{i\omega})\, \overline{e^{i\omega_0}\Omega_q(e^{i\omega_0})}\{w(e^{-i\omega})\, \overline{w(e^{-i\omega_0})}\}^{n-1}$$
$$-\Omega_q(e^{-i\omega})\, \overline{\Omega_q(e^{-i\omega_0})}\{w(e^{i\omega})\overline{w(e^{i\omega_0})}\}^{n-1}$$

$$\sin\frac{\alpha}{2} F_2(e^{i\omega}) \stackrel{\text{def}}{=} e^{i\omega}\Omega_q(e^{i\omega})\, \overline{\Omega_q(e^{-i\omega_0})}\{w(e^{-i\omega})\, \overline{w(e^{i\omega_0})}\}^{n-1}$$
$$-\Omega_q(e^{-i\omega})\, \overline{e^{i\omega_0}\Omega_q(e^{i\omega_0})}\{w(e^{i\omega})\, \overline{w(e^{-i\omega_0})}\}^{n-1}.$$

By the change of variables formula (19) we have

$$I_n \leqslant C_{15} \int_0^{\pi} \left\{ \frac{|F_1(e^{i\omega})|}{|e^{i(\omega-\omega_0)}-1|} + \frac{|F_2(e^{i\omega})|}{|e^{i\omega}-e^{i\omega_0}|} \right\} d\omega$$
$$= C_{15} \int_0^{\pi} \frac{|F_1(e^{i\omega})| + |F_2(e^{i\omega})|}{|e^{i\omega}-e^{i\omega_0}|}\, d\omega. \tag{90}$$

The rest is standard, if we take into account that for $j = 1, 2$

$$F_j(e^{i\omega_0}) = 0, \qquad |F_j(v)| = O(1), \qquad |F_j'(v)| = O(n), \qquad n \to \infty$$

uniformly on the unit circle (cf. (62) and (86)). Indeed,

$$\int_0^{\pi} \frac{|F_j(e^{i\omega})|}{|e^{i\omega}-e^{i\omega_0}|}\, d\omega \leqslant \int_{|\omega-\omega_0|\leqslant 1/n} O(n)\, d\omega + \int_{|\omega-\omega|>1/n} \frac{O(1)\, d\omega}{|e^{i\omega}-e^{i\omega_0}|}$$
$$\leqslant C_{16}\left(1 + \log\frac{1}{n}\right).$$

Thus (89) takes the form

$$\left| \varphi_n(e^{i\vartheta_0}) - \frac{\kappa_n}{\kappa_n(\Omega_q)}\, \varphi_n(e^{i\vartheta_0}; \Omega_q) \right| \leqslant C_{16} M_n \omega\left(\frac{1}{n}, \rho\right)\left(1 + \log\frac{1}{n}\right). \tag{91}$$

The Dini condition (87) now appears on the scene

$$M_n\left(1-C_{16}\omega\left(\frac{1}{n},\rho\right)\left(1+\log\frac{1}{n}\right)\right)\leqslant C_{17},$$

which yields (88). ▮

The following result, which serves to connect two orthogonal polynomials systems on $\Delta_\alpha$, is a direct consequence of (91), (88), and (72).

Corollary 12. *Under the Dini condition* (87) *the limit relation*

$$\lim_{n\to\infty}(\varphi_n(e^{i\vartheta})-\varphi_n(e^{i\vartheta};\Omega_q))=0 \tag{92}$$

*holds uniformly on the arc* $\Delta_\alpha$.

To obtain the asymptotic representation for orthonormal polynomials $\varphi_n$ on $\Delta_\alpha$, a somewhat more restrictive assumption on the function $\rho$, than (87), is required.[2] We call it the *Zygmund condition*:

$$\int_0^{\pi-\alpha}\frac{\omega(x,\rho)}{x}\,dx<\infty. \tag{93}$$

A simple inequality

$$\frac{1}{2}\,\omega(t,\rho)\log\frac{1}{t}\leqslant\int_t^{\sqrt{t}}\frac{\omega(x,\rho)}{x}\,dx=o(1),\qquad t\to 0$$

displays that (93) implies (87).

In light of known expression (47) for Akhiezer's polynomials on the arc $\Delta_\alpha$ Proposition 11 gives rise to the asymptotic formula for orthonormal polynomials $\varphi_n$ on $\Delta_\alpha$. In fact, we need only to prove that

$$\lim_{n\to\infty} g_\pm(e^{i\vartheta_1};\rho_q)=g_\pm(e^{i\vartheta_1};\rho) \tag{94}$$

uniformly on $\Delta_\alpha$. By (35) and (63), it suffices to show that

$$\lim_{n\to\infty}\ \text{v.p.}\int_\alpha^{2\pi-\alpha}\left(\log\frac{1}{\rho(e^{i\vartheta})}-\log\frac{1}{\rho_q(e^{i\vartheta})}\right)\times\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}=0,$$

[2] It is known (cf [7]) that in the case of the whole unit circle the Dini condition itself does not guarantee the uniform asymptotic representation for orthonormal polynomials.

or, in other words,

$$\lim_{n\to\infty}\int_{\alpha}^{2\pi-\alpha}\left(\log\frac{\rho(e^{i\vartheta_1})}{\rho(e^{i\vartheta})}-\log\frac{\rho_q(e^{i\vartheta_1})}{\rho_q(e^{i\vartheta})}\right)\times\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}=0 \tag{95}$$

uniformly on $\Delta_\alpha$. Without loss of generality, we may assume that $\alpha\leqslant\vartheta_1\leqslant\pi$ and that the integral in (95) is taken over the interval $\alpha\leqslant\vartheta\leqslant\pi+\alpha_1$, $\alpha_1=(\pi-\alpha)/2$ (the rest of the integral tends to zero automatically). Note that under such assumptions for the kernel function in the right hand side of (95) a double inequality

$$C_{18}\sqrt{\frac{\vartheta_1-\alpha}{\vartheta-\alpha}}\leqslant\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_2/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\leqslant C_{19}\sqrt{\frac{\vartheta_1-\alpha}{\vartheta-\alpha}}$$

holds.

To prove (95), we proceed in two steps. The first one, which concerns the integral off a vicinity of the point $\vartheta_1$, is plain. The second one, which deals with the vicinity of $\vartheta_1$, is a little more elaborate.

*Step* 1. We have

$$\begin{aligned}&\int_{|\vartheta_1-\vartheta|>n^{-3}}\left(\log\frac{\rho(e^{i\vartheta_1})}{\rho(e^{i\vartheta})}-\log\frac{\rho_q(e^{i\vartheta_1})}{\rho_q(e^{i\vartheta})}\right)\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}\\&\quad\leqslant C_{20}\,\omega\left(\frac{1}{n},\rho\right)\int_{|\vartheta_1-\vartheta|>n^{-3}}\sqrt{\frac{\vartheta_1-\alpha}{\vartheta-\alpha}}\,\frac{d\vartheta}{|\vartheta-\vartheta_1|}\\&\quad\leqslant C_{21}\,\omega\left(\frac{1}{n},\rho\right)\left(1+\log\frac{1}{n}\right)=o(1),\qquad n\to\infty.\end{aligned}$$

*Step* 2. Let us rearrange the terms in the left hand side of (95) and show that

$$\lim_{n\to\infty}\int_{|\vartheta-\vartheta_1|\leqslant n^{-3}}\left(\log\frac{1}{\rho(e^{i\vartheta_1})}-\log\frac{1}{\rho(e^{i\vartheta})}\right)\times\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}=0, \tag{96}$$

$$\lim_{n\to\infty}\int_{|\vartheta-\vartheta_1|\leqslant n^{-3}}\left(\log\frac{1}{\rho_q(e^{i\vartheta_1})}-\log\frac{1}{\rho_q(e^{i\vartheta})}\right)\times\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}=0. \tag{97}$$

To prove (96), we use the relation

$$\left|\log\frac{1}{\rho(e^{i\vartheta_1})}-\log\frac{1}{\rho(e^{i\vartheta})}\right|\leqslant C_{22}\omega(|\vartheta-\vartheta_1|,\rho), \tag{98}$$

which is a direct consequence of (49). Let $\vartheta_{n,1}\overset{\text{def}}{=}\max(\vartheta_1-n^{-3},\alpha)$. Then,

$$\begin{aligned}
&\left|\int_{\vartheta_{n,1}}^{\vartheta_1}\left(\log\frac{1}{\rho(e^{i\vartheta_1})}-\log\frac{1}{\rho(e^{i\vartheta})}\right)\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}\right|\\
&\quad\leqslant C_{23}\int_{\vartheta_{n,1}}^{\vartheta_1}\frac{\omega(\vartheta_1-\vartheta,\rho)}{\vartheta_1-\vartheta}\sqrt{1+\frac{\vartheta_1-\vartheta}{\vartheta-\alpha}}\,d\vartheta\\
&\quad\leqslant C_{23}\int_{\vartheta_{n,1}}^{\vartheta_1}\frac{\omega(\vartheta_1-\vartheta,\rho)}{\vartheta_1-\vartheta}\,d\vartheta+C_{23}\int_{\vartheta_{n,1}}^{\vartheta_1}\frac{\omega(\vartheta_1-\vartheta,\rho)}{\sqrt{(\vartheta_1-\vartheta)(\vartheta-\alpha)}}\,d\vartheta\\
&\quad=C_{23}\int_0^{\vartheta_1-\vartheta_{n,1}}\frac{\omega(x,\rho)}{x}\,dx+C_{23}\int_0^{\vartheta_1-\vartheta_{n,1}}\frac{\omega(x,\rho)}{\sqrt{x(\vartheta_1-\alpha-x)}}\,dx\\
&\quad=I_1+I_2.
\end{aligned}$$

Since $0\leqslant\vartheta_1-\vartheta_{n,1}\leqslant n^{-3}$, we see that

$$I_1\leqslant C_{23}\int_0^{n^{-3}}\frac{\omega(x,\rho)}{x}\,dx,\qquad n\to\infty.$$

Next, if $\vartheta_{n,1}=\alpha\geqslant\vartheta_1-n^{-3}$, the, putting $b=\vartheta_1-\alpha\leqslant n^{-3}$, we get

$$\begin{aligned}
I_2&=C_{23}\int_0^b\frac{\omega(x,\rho)}{\sqrt{x(b-x)}}\,dx=C_{23}\int_0^1\frac{\omega(by,\rho)}{\sqrt{y(1-y)}}\,dy\\
&\leqslant C_{24}\omega(b,\rho)\leqslant C_{24}\omega(n^{-3},\rho).
\end{aligned}$$

If, on the other hand, $\vartheta_{n,1}=\vartheta_1-n^{-3}\geqslant\alpha$, then

$$I_2=C_{23}\int_0^{n^{-3}}\frac{\omega(x,\rho)}{\sqrt{x(b-x)}}\,dx=C_{23}\int_0^1\frac{\omega(n^{-3}y,\rho)}{\sqrt{y(1-y)}}\,dy\leqslant C_{24}\omega(n^{-3},\rho).$$

Hence by Zygmund's condition

$$\begin{aligned}
&\left|\int_{\vartheta_{n,1}}^{\vartheta_1}\left(\log\frac{1}{\rho(e^{i\vartheta_1})}-\log\frac{1}{\rho(e^{i\vartheta})}\right)\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}\right|\\
&\quad\leqslant C_{24}\left\{\int_0^{n^{-3}}\frac{\omega(x,\rho)}{x}\,x+\omega(n^{-3},\rho)\right\}=o(1).
\end{aligned}$$

For the second part of the integral in (96) we have

$$\left|\int_{\vartheta_1}^{\vartheta_1+n^{-3}}\left(\log\frac{1}{\rho(e^{i\vartheta_1})}-\log\frac{1}{\rho(e^{i\vartheta})}\right)\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}\right|$$
$$\leqslant C_{24}\int_0^{n^{-3}}\frac{\omega(x,\rho)}{x}\,dx,$$

and thus (96) is verified.

Turning to (97) we shall establish first an inequality, which is similar to (98). The relations (64) and (62) imply

$$\begin{aligned}
&\left|\frac{1}{\rho_q(e^{i\vartheta_1})}-\frac{1}{\rho_q(e^{i\vartheta})}\right|\\
&\quad=\frac{2}{\sin(\alpha/2)}\left|\,|\Omega_q(e^{i\omega})|^2\sin^2\frac{\vartheta}{2}-|\Omega_q(e^{i\omega_1})|^2\sin^2\frac{\vartheta_1}{2}\right|\\
&\quad\leqslant\frac{2}{\sin(\alpha/2)}\left(|\Omega_q(e^{i\omega})|^2\left|\sin^2\frac{\vartheta}{2}-\sin^2\frac{\vartheta_1}{2}\right|\right.\\
&\qquad\left.+\sin^2\frac{\vartheta_1}{2}\,\big|\,|\Omega_q(e^{i\omega})|^2-|\Omega_q(e^{i\omega_1})|^2\big|\right)\\
&\quad\leqslant C_{25}(|\vartheta-\vartheta_1|+|\Omega_q(e^{i\omega})-\Omega_q(e^{i\omega_1})|)\\
&\quad\leqslant C_{25}(|\vartheta-\vartheta_1|+\max|\Omega_q'|\,|e^{i\omega}-e^{i\omega_1}|).
\end{aligned}$$

Next, it is not hard to deduce from (9) that

$$|e^{i\omega}-e^{i\omega_1}|\leqslant C_{26}\sqrt{|\vartheta-\vartheta_1|}.$$

Thus, thanks to (86), we come to the conclusion (cf. (98))

$$\left|\log\frac{1}{\rho_q(e^{i\vartheta_1})}-\log\frac{1}{\rho_q(e^{i\vartheta})}\right|\leqslant C_{27}n\sqrt{|\vartheta_1-\vartheta|}. \tag{99}$$

Further calculations in much the same way as above in Step 1 lead to the bound

$$\int_{|\vartheta-\vartheta_1|\leqslant n^{-3}}\left(\log\frac{1}{\rho_q(e^{i\vartheta_1})}-\log\frac{1}{\rho_q(e^{i\vartheta})}\right)$$
$$\times\sqrt{\frac{\cos^2(\alpha/2)-\cos^2(\vartheta_1/2)}{\cos^2(\alpha/2)-\cos^2(\vartheta/2)}}\,\frac{d\vartheta}{\sin((\vartheta-\vartheta_1)/2)}\leqslant C_{28}n^{-1/2},$$

which completes the proof of (97).

We are now in a position to sum up the results, obtained above, in the following statement.

THEOREM 13. *Let the weight function $W$ satisfy* (48), (49), *and Zygmund's condition* (93). *Then for the orthonormal polynomials $\varphi_n$ the asymptotic representation*

$$\varphi_n(e^{i\vartheta}) = \frac{e^{-i\lambda}\sqrt{1-\sin(\alpha/2)} - e^{-i(\vartheta/2)}\sqrt{1+\sin(\alpha/2)}}{2i\sin(\vartheta/2)} \times \exp\left\{in\left(\frac{\vartheta}{2}-\lambda\right)\right\} g_-(e^{i\vartheta};\rho) + \frac{e^{i\lambda}\sqrt{1-\sin(\alpha/2)} - e^{-i(\vartheta/2)}\sqrt{1+\sin(\alpha/2)}}{2i\sin(\vartheta/2)} \times \exp\left\{in\left(\frac{\vartheta}{2}+\lambda\right)\right\} g_+(e^{i\vartheta};\rho) + o(1)$$

*holds uniformly on the arc $\Delta_\alpha$.*

## REFERENCES


1. N. I. Akhiezer, On polynomials orthogonal on a circular arc, *Dokl. Akad. Nauk SSSR* **130** (1960), 247–250 [in Russian]; *Soviet Math. Dokl.* **1** (1960), 31–34.
2. M. Bello Hernández and G. López Lagomasino, Ratio and relative asymptotics of polynomials orthogonal on an arc of the unit circle, *J. Approx. Theory*, in press.
3. S. N. Bernstein, Sur la limitation des dérivées des polynômes, *Comptes Rendus* **190** (1930), 338–341.
4. Ya. L. Geronimus, Polynomials orthogonal on a circle and their applications, *in* "Series and Approximations," Amer. Math. Soc. Transl. Ser. 1, Vol. 3, pp. 1–78, Amer. Math. Soc., Providence, 1962.
5. Ya. L. Geronimus, Orthogonal polynomials, *in* "Two Papers on Special Functions," Amer. Math. Soc. Transl. Ser. 2, Vol. 108, pp. 37–130, Amer. Math. Soc., Providence, 1977.
6. B. L. Golinskii, Asymptotic representation of orthogonal polynomials, *Uspekhi Mat. Nauk*. **35** (1980), 145–196. [in Russian]
7. B. L. Golinskii, On two basic conditions for asymptotic representation of orthonormal polynomials on the unit circle, *Mat. Zametki* **15** (1974), 847–855. [in Russian]
8. U. Grenander and G. Szegő, "Toeplitz Forms and Their Applications," Univ. California Press, Berkeley, 1958; 2nd ed., Chelsea, New York, 1984.
9. V. A. Kaliaguine, A note on the asymptotics of orthogonal polynomials on a complex arc: The case of a measure with a discrete part, *J. Approx. Theory* **80** (1995), 138–145.
10. A. Máté, P. Nevai, and V. Totik, Extensions of Szegő's theory of orthogonal polynomials, II, *Constr. Approx.* **3** (1987), 51–72.
11. A. Máté, P. Nevai, and V. Totik, Extensions of Szegő's theory of orthogonal polynomials, III, *Constr. Approx.* **3** (1987), 73–96.

12. F. Peherstorfer and R. Steinbauer, Asymptotic behavior of orthogonal polynomials on the unit circle with asymptotically periodic recurrence coefficients, *J. Approx. Theory* **88** (1997), 316–353.
13. E. A. Rahmanov, On the asymptotics of the ratio of orthogonal polynomials, II, *Math. USSR-Sb.* **46** (1983), 105–117; Russian original; *Mat. Sb.* **118**, No. 160 (1982), 104–117.
14. V. I. Smirnov and N. A. Lebedev, "Functions of a Complex Variable, Constructive Theory," MIT Press, Cambridge, MA, 1968.
15. G. Szegő, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, 1975.
16. H. Widom, Extremal polynomials associated with a system of curves and arcs in the complex plane, *Adv. Math.* **3** (1969), 127–232.